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ASYMPTOTIC DESCRIPTION OF VORTEX FILAMENTS IN AN INCOMPRESSIBLE FLUID†

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General equations which describe vortex filaments in an incompressible, low viscosity fluid are derived. Perturbation theory, which leads to these equations, is based on the topological properties of the trajectories of steady flows of an ideal fluid. The vortex filament equations turn out to be similar to the Prandtl equations in boundary-layer theory and the equations of V. P. Maslov, which describe periodic "coherent" pulsations. An infinite series of integral relations, which, when the viscosity disappears, reduce to conservation laws, is found in the case of these equations. It is shown that the well known Moffatt–Kida–Ohkitani vortex is the simplest "steady" solution of the equations of a vortex filament. The equations of an elongated vortex are naturally treated as equations which are defined in a graph with boundary conditions at its vertices. The graph which occurs in this case is found to be associated in a natural way with the Morse theory and the topological theory of integrable Hamiltonian systems and is identical to the Reef and Fomenko invariants which are well known in these theories.

From a mathematical point of view, localized perturbations in an incompressible fluid (vortex filaments, rings, films, narrow traces and nappes, boundary layers, etc.) are solutions of the equations of hydrodynamics containing a small parameter, the characteristic width of the perturbation compared with the scale of the external flow. Asymptotic methods, based on the deformation of some family of exact solutions of model (unperturbed) problems, are therefore used to describe them.

An asymptotic scheme is developed below which describes vortex filaments (that is, perturbations concentrated in the small neighbourhood of a curve in three-dimensional space) and which uses the solutions of the steady-state Euler equations for an ideal fluid as the "zeroth approximation": it is therefore assumed that the viscosity is fairly low. It is well known (see [1, 2], for example) that Eulerian flows possess rich topological properties and, moreover, the families of such flows are apparently parametrized by the topological characteristics of their trajectories ([3, 1]). This is largely reflected in perturbation theory, and, in particular, the fundamental equations of the asymptotic scheme, that is, the equations of a vortex filament, are found to be closely associated with the topological characteristics of the trajectories of the corresponding Eulerian field. The central result of this paper is the derivation of the above-mentioned equations and their investigation. An analogy is established between the equations of an elongated vortex and the Prandtl equations (in particular, equations with a self-induced pressure). Conservation laws and integral identities, which the motion in a vortex filament satisfies, are also presented.

1. PERTURBATION THEORY IN THE PROBLEM OF A VORTEX FILAMENT

The steady-state three-dimensional velocity field, $u(x)$ of an incompressible, viscous fluid satisfies the Navier–Stokes equations

$$(u, \nabla)u + \nabla p = \nu\Delta u \quad (\nabla, u) = 0 \tag{1.1}$$

Suppose a smooth, external steady flow of a fluid is specified, that is, a solution $\mathbf{V}(x, \nu)$, $\mathbf{P}(x, \nu)$ of these equations which is bounded together with all of it derivatives and depends smoothly on its arguments $x \in R^3$, $\nu \in [0, \infty]$. We shall consider a perturbation of this flow which is localized in a small neighbourhood of a certain (previously unknown) curve γ in three-dimensional space. The characteristic width, ε, of this neighbourhood (compared with the scale of the external field V) will be a small parameter in the problem under consideration. It is natural to treat this perturbation as a vortex

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filament located in an external flow and, here, there is no assumption regarding the smallness of the perturbation compared with the field **V**. It can be shown (see [4], for example, where this was done for periodic coherent structures) that such rapidly changing solutions can only exist if the viscosity is sufficiently low or, more accurately, if $\nu = O(\varepsilon^2)$. Otherwise, the viscosity destroys the vortex. We shall therefore henceforth assume that $\nu = \varepsilon^2\nu_0$, $\nu_0 = O(1)$.

Suppose $x = R(s)$ are the parametric equations of the curve γ, s is the length of an arc and we denote the orthonormalized n-hedron in the plane normal to γ which satisfies the condition $(n_1(s), n_2(s)) = 0$ by $n_1(s)$, $n_2(s)$. We now consider a certain neighbourhood G' of the curve γ which is independent of ε (but sufficiently small) and introduce the coordinates $\{s, y_1, y_2\}$ in this neighbourhood using the formulae

$$x = R(s) + y_1 n_1(s) + y_2 n_2(s)$$

The solution u, p of the three-dimensional Navier–Stokes equations in a smaller neighbourhood $G \subset G'$ will be sought in the form

$$u = \mathbf{V}(x, \varepsilon) + \mathbf{U}(y/\varepsilon, x, \varepsilon), \qquad p = \mathbf{P}(x, \varepsilon) + \pi(y/\varepsilon, x, \varepsilon) \tag{1.2}$$

where the functions $\mathbf{U}(z, x, \varepsilon)$, $\pi(z, x, \varepsilon)$ depend smoothly on all of their arguments when $\varepsilon \in (0, \varepsilon_0)$, $z \in R^2$, $x \in G$ and $\mathbf{U}(z, x, \varepsilon)$, $\pi(z, x, \varepsilon) \to 0$ when $|z| \to \infty$ not more slowly than $O(|z|^{-2})$. The two-dimensional vector of the "stretched" coordinates in the plane which is normal to γ will henceforth be denoted by $\gamma : z = y/\varepsilon$. Outside G, the functions **U**, π are multiplied by a smooth cutoff function which is equal to zero outside G'.

We shall treat the problem of a vortex filament locally in a certain bounded open domain without being concerned about the boundary conditions or the conditions at infinity. As in the case of well-known boundary layer problems (see [5], for example), in the case of the present problem it is possible to consider "the problem of the continuation of a vortex" by assuming that a field of the form (1.2) is specified in a certain surface M which transversely intersects the curve γ. However, unlike the problem of the continuation of a boundary layer, this "initial" field cannot be specified arbitrarily and it follows from the results of the following section that it must satisfy Eqs (1.5)–(1.6) which are obtained below with respect to the "fast" variable z. This situation is characteristic in the case of the asymptotic solutions of non-linear equations (cf. [6–8]) and, as a rule, in such problems the form of the leading part of the asymptotic form is by no means arbitrary.

We expand the functions from (1.2) using Taylor's formula when $\varepsilon \to 0$

$$\begin{gathered}\mathbf{V}(x, \varepsilon) = V(x) + \varepsilon^2 V_1(x) + \ldots, \quad \mathbf{P}(x, \varepsilon) = P(x) + \varepsilon^2 P_1(x) + \ldots \\ \mathbf{U}(z, x, \varepsilon) = U(z, s) + \varepsilon U_1(z, x) + \ldots, \quad \pi(z, x, \varepsilon) = \pi_0(z, s) + \varepsilon \pi_1(z, x) + \ldots\end{gathered} \tag{1.3}$$

and we substitute these expansions into the Navier–Stokes equations (1.1).

We will first consider the equalities which have been obtained outside a neighbourhood of the curve γ, which may be as small as desired (that is, when $|z| \to \infty$). On equating the coefficient of ε_0 to zero, we obtain the three-dimensional Euler equations for the vector field $V(x)$

$$(V, \nabla)V + \nabla P = 0 \quad (\nabla, V) = 0 \tag{1.4}$$

Subsequently, it is assumed everywhere that $V(x)$ is the smooth solution of these equations. Within the domain G, we write the Navier–Stokes equations in (y, s) coordinates, substitute expressions (1.3) into Eqs (1.1), put $y_j = \varepsilon z_j$, expand the resulting relations using Taylor's formula when $\varepsilon \to 0$ and equate the coefficients of all power of ε to zero. For ε^{-1}, we obtain

$$(\upsilon, \nabla_z)\upsilon + \nabla_z \pi_0 = 0 \quad (\nabla_z, \upsilon) = 0 \tag{1.5}$$

$$(\upsilon, \nabla_z)w = 0 \tag{1.6}$$

$$\nabla_z = (\partial / \partial z_1, \ \partial / \partial z_2), \quad \upsilon_j = (V|_\gamma + U, \ n_j), \quad w = (V|_\gamma + U, dR/ds)$$

Henceforth, we shall consider the solutions υ of the Euler equations in the plane of the "fast" variables z which satisfy the following conditions:

1. the vector field υ only has a finite number of singular points, and all of them are non-degenerate (then, by virtue of the solenoidal character of υ, these singular points are either saddle points or centres) and just one singular point lies in each separatrix;

2. almost all of the trajectories of the field v are closed;

It follows from the second requirement and from the fact that $|U| \to 0$ when $|z| \to \infty$ that the vector field v decays at infinity and, therefore, $(|V|_\gamma, n_j) = 0$, that is, the curve γ is a trajectory of the field $V(x)$. The following assertion is thereby proved.

Assertion 1. Suppose functions (1.2) satisfy mod $O(1)$ the Navier–Stokes equations (1.1). Then,

(a) the vector field $v(z)$ satisfies the two-dimensional Euler equations in the plane;

(b) the function w is constant in the trajectories of the field v;

(c) moreover, if conditions 1 and 2 are satisfied, the curve γ is a trajectory of the vector field $V(x)$.

We will write out the equations obtained by equating to zero the coefficient of ε^0, which arises when functions (1.2) are substituted into the Navier–Stokes equations (1.1). We get

$$\begin{aligned} &-(v,\ \nabla_z)w_1 - (v^1, \nabla_z)w = H(z,s) \\ &-(v,\ \nabla_z)v^1 - (v^1,\nabla_z)v - \nabla_z\pi_1 = F(z,s); \quad -(\nabla_z,\ v^1) = G(z,s) \end{aligned} \tag{1.7}$$

Here

$$\begin{aligned} &H(z,s) = w\frac{\partial w}{\partial s} - \frac{1}{2}\frac{\partial}{\partial s}W^2 - k(s)(e(s),v)w + (b,\ v) + (Az,\nabla_z)w + \frac{\partial \pi_0}{\partial s} - \nu_0\Delta_z w \\ &F(z,s) = w\frac{\partial v}{\partial s} + k(s)e(s)(w^2 - W^2) + Av + (Az,\nabla_z)v - \nu_0\Delta_z v \\ &G(z,s) = \frac{\partial}{\partial s}(W - w) - k(s)(e(s),v) \\ &v_j^1 = (U_1, n_j), \quad w_1 = (U_1,\ R'(s)), \quad W(s) = (V, R'(s))|_\gamma \\ &b_j(s) = (R'(s),\ \partial V/\partial y_j|_\gamma), \quad A_{ij}(s) = (n_i, \partial V/\partial y_j|_\gamma) \end{aligned}$$

This system consists of the linearized equations (1.5)–(1.6) with non-zero right-hand sides. The dependence of the field v and the function w on the "slow" variable s is determined from the conditions for this system to be solvable (compare with [6–9]).

Before writing out these conditions, we present an assertion concerning the cokernel of the linearized Euler operator which has previously been proved [10, 11]. We consider equations which are formally adjoint to the linearized equations (1.5) in the space of solenoidal vector fields

$$(v,\nabla_z)\xi - \frac{\partial v}{\partial z}^{*}\xi = \nabla_z\chi, \quad (\nabla_z,\xi) = 0 \tag{1.8}$$

Here ξ is a two-dimensional vector field and χ is a scalar function.

Theorem 1. Suppose $v(z)$ is a smooth of Eqs (1.5) with properties 1 and 2. Then, any smooth vector field ξ which commutes with the field $v(z)$ satisfies Eqs (1.8).

Remark. Vector fields which commute with the field v obviously form an infinite-dimensional linear space (cf. [4, 12]) and, in particular, all fields of the form $\{-\partial f/\partial z_2, \partial f/\partial z_1\}$, where $f(z)$ is an arbitrary smooth function which is constant in the trajectories of the field v, are contained in this space.

We will now present the conditions for the system of equations (1.7) to be solvable (which have also been obtained previously in [10, 11]).

Theorem 2. Suppose a smooth solution (v^1, w_1, π_1) of system (1.7) exists. Then, its right-hand sides $H(z, s)$, $F(z, s)$, $G(z, s)$, satisfy the equalities

$$\begin{aligned} &\frac{1}{2\pi}\oint(F,v)d\varphi + a_0(I,s)\frac{\partial B_0}{\partial I} = 0, \quad B_0 = \frac{v^2}{2} + \pi_0 \\ &\frac{1}{2\pi}\oint G d\varphi + \frac{\partial a_0}{\partial I} = 0, \quad \frac{1}{2\pi}\oint H d\varphi + a_0(I,s)\frac{\partial w}{\partial I} = 0 \end{aligned} \tag{1.9}$$

Here (and, subsequently, in Section 2) integration is carried out along an arbitrary closed trajectory of the vector field $v(z)$, (I, φ) are the action-angle of the field v which are defined in the neighbourhood of this trajectory (see below) and $a_0(I, s)$ is an auxiliary function.

Remark. The above conditions arise from the conditions for the right-hand sides of system (1.7) to be orthogonal to the space from the cokernel of the Euler operator indicated in Theorem 1 (together with the obvious infinite-dimensional cokernel of the linear operator $(v, \nabla z)$; the latter consists of functions which are constant in trajectories of the field v). A "basis" of this space is selected in writing the orthogonality conditions in a special manner. Roughly speaking, it consists of δ-shaped fields with carriers in the trajectories of v. In fact, such a choice of "basis" enables one to reduce the problem of finding the orthogonality conditions to averaging along the trajectories of the field v.

2. THE EQUATIONS OF A VORTEX FILAMENT AND THEIR RELATION TO THE TOPOLOGICAL INVARIANTS OF THE VECTOR FIELDS

Certain topological and hydrodynamic characteristics of the two-dimensional vector field $v(z)$ (which depends on s as well as on a parameter) are required in the subsequent treatment. Actually, we consider an arbitrary domain in the $\{z_1, z_2\}$ plane in which there are no equilibrium points and separatrices of the field v. A closed trajectory of this field passes through each point of this domain, and we denote the area of the domain which is bounded by a trajectory, divided by 2π, by I. The coordinates (the action-angle variables (see [1], for example)) I and φ can be introduced in the domain being considered such that $\varphi \in [0, 2\pi]$ is the angular variable in a closed trajectory, an element of volume has the form $dI \wedge d\varphi$ and the field $v = \omega(I, s)\partial/\partial\varphi$, where ω is the frequency. Functions, which are constant on the trajectories of the field v, have the form $f(I, s)$ (they are independent of φ). In particular, it follows from Eq. (1.6) that the function w is of this form. Furthermore, it is well known that the Bernoulli integral $v^2/2 + \pi_0$ of an Eulerian field is also constant on the trajectories. The function $v^2/2 + \pi_0 + w^2/2$ is denoted by $B(I, s)$. Hence, two characteristics of a vortex filament are defined in each domain which does not contain singular points and the separatrices of the field v and these are functions of the two variables $B(I, s)$ and $w(I, s)$.

Theorem 3 [10, 11]. Equalities (1.9) are equivalent to the following system of equations

$$w\frac{\partial w}{\partial s} + a\frac{\partial w}{\partial I} + \langle \pi_s \rangle = \nu_0 \frac{\partial}{\partial I}\left(D^2 \frac{\partial w}{\partial I}\right)$$

$$\frac{\partial w}{\partial s} + \frac{\partial a}{\partial I} = 0; \quad w\frac{\partial B}{\partial s} + a\frac{\partial B}{\partial I} = \nu_0\left(\frac{\partial}{\partial I}\left(D^2\frac{\partial B}{\partial I}\right) - \Lambda^2\right) \tag{2.1}$$

where

$$a = a_0 - I\frac{\partial W}{\partial s}, \quad \langle \pi_s \rangle = \frac{1}{2\pi}\oint \frac{\partial}{\partial s}\left(\pi_0 - \frac{1}{2}W^2\right)d\varphi$$

$$D^2 = \frac{1}{2\pi}\oint\left(\frac{\partial z}{\partial \varphi}\right)^2 d\varphi, \quad \Lambda^2 = \left(\frac{\partial v_2}{\partial z_1} - \frac{\partial v_1}{\partial z_2}\right)^2 + D^2\left(\frac{\partial w}{\partial I}\right)^2$$

This system is a set of equations in the functions w and B, defined in different domains of the z plane which do not contain separatrices. In each domain, the variable I changes in a segment (or in a half line, if the domain contains an infinitely distant point). Each separatrix of a saddle point of v has the form of a figure eight and separates three such domains, and the values of the variable I at the ends of the corresponding segments, corresponding to a given separatrix, are connected by the relation $I_3 = I_1 + I_2$, where I_1, I_2 are the areas (divided by 2π) under the two parts of the separatrix into which it is separated by the saddle point, and I_3 is the area under the whole of the separatrix. Since the solution of the Navier–Stokes equations is assumed to depend continuously on the fast variables z, each of the functions w and B takes the same value at the points I_1, I_2, I_3 which correspond to a given separatrix and this value is equal to the value of the corresponding function in the saddle. Hence, matching conditions arise at the ends of the segments in which Eqs (2.1) are defined.

In order to provide a clear interpretation of the relations obtained, we consider a set of segments in a plane, each of which corresponds to a certain domain of variation of the variable I, and put the ends of the segments corresponding to a single separatrix at one point of the plane. As a result, we obtain the graph Γ which is a binary tree and either three edges join at each vertex (it then corresponds to a

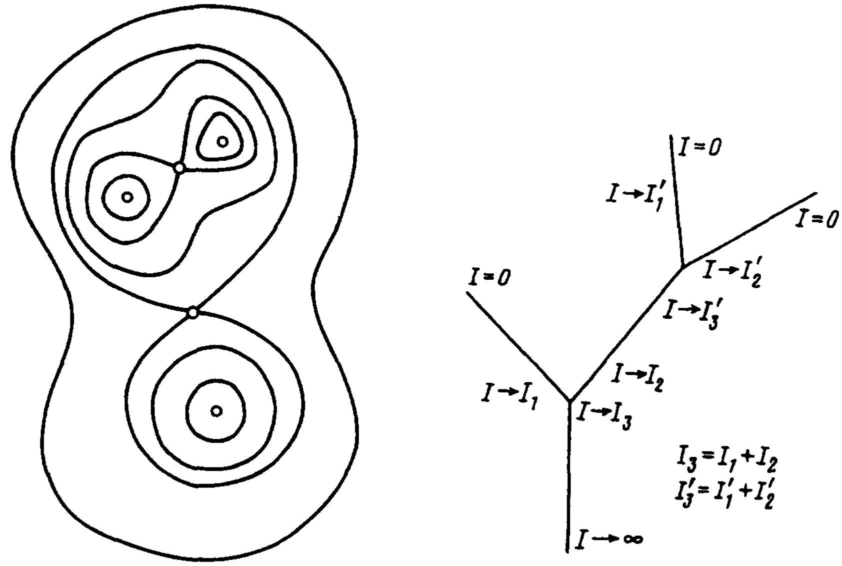

Fig. 1.

separatrix of a saddle of the field υ) or one edge terminates (this vertex then corresponds to an equilibrium position of the "centre" type), and the graph does not contain closed paths (cycles). An example of a phase portrait of the field υ and the graph Γ corresponding to it are shown in Fig. 1.

The equations of a vortex filament in the variable I are defined on this graph and the functions B and w are continuous on it. Note that the function a does not possess the property of continuity. It has been shown [10] that, if Eqs (1.7) are solvable, then $a = 0$ at all vertices of degree 1 and the Kirchhoff condition in the theory of electric circuits: $a_3 = a_1 + a_2$, where $a_j = a(I_j)$, is satisfied at each vertex of degree 3.

The structure of Eqs (2.1) and the additional conditions imposed on the function a is not random and is associated with the topological properties of the solutions of the steady-state Euler equations in the following way. The problem of finding the solutions of Eqs (1.5), with a topology of the trajectories specified in advance, has been discussed previously ([3, 1, etc.]). The corresponding procedure has been described most fully by Moffatt [3] in the context of the theory of magnetic relaxation he has developed. This scheme is based on the properties of the equations of the magnetohydrodynamics (MHD) of an ideally conducting fluid, which have the form

$$\frac{\partial u}{\partial t} + (u, \nabla_z)u + \nabla_z p = \mu\Delta_z u - H \times \operatorname{rot}_z H$$

$$\frac{\partial H}{\partial t} + \{u, H\} = 0 \quad (\nabla_z, u) = (\nabla_z, H) = 0$$

where $H(z, t)$ is the magnetic field, $u(z, t)$ is the velocity field of the conducting fluid, $\{,\}$ is the commutator of the vector fields and μ is the coefficient of viscosity (this coefficient is assumed to be quite large in Moffatt's case).

The idea behind Moffatt's scheme is as follows. Consider the Cauchy problem $u|_{t=0} = 0, H|_{t=0} = H_0(z)$, $(\nabla_z, H_0) = 0$ for the system of equations which has been written. From physical considerations, it is natural to assume that the velocity of the fluid decays with time, that is, $u(z, t) \to 0$ when $t \to \infty$. It then follows from the second equation of the MHD system that, when $t \to \infty$, the vector field H (the magnetic field) tends to a certain limiting field $H_\infty(z)$. On the other hand, we obtain from the first equation at the same time that H_∞ satisfies the equation $H_\infty \times \operatorname{rot}_z H_\infty = -\nabla_z p$, which is simply another form of writing down the steady-state Euler equations (1.5). So, for long times, the magnetic field H, while evolving by virtue of the MHD equations, tends to the solution of the Euler equations. On the other hand, the second equation of the MHD system signifies that the vector field $H(z, t)$ is obtained from the vector field $H_0(z)$ by a translation of the field u along the flow, that is (since this field is solenoidal), by using a transformation of the z coordinate which preserves the element of area.

It follows from Moffatt's scheme that the families of solutions of Eqs (1.5) must be parametrized by the characteristics of the field v, which are invariant under changes in the coordinates with a unit Jacobian (by topological invariants). These invariants can be described in explicit form: they consist of a set of trajectories of the field v, which is identical to the parametrized graph Γ, introduced above, and a continuous function (of frequency, for example) in it. Furthermore, the set of solutions of Eqs (1.6) is a set of functions which are constant in the trajectories of v, that is, defined in the same graph. Hence, the parameters, on which the solutions of system (1.5), (1.6) depend, are the pair of functions on the graph which can evolve along the axis of the vortex, that is, they can still depend on the variable s.

Theorem 1 asserts that the space of the functions on the graph Γ is contained in the set of solutions of the equations which are adjoint to the linearized Euler equations. Actually, a solenoidal vector field which commutes with v has the same trajectories as v and differs from v solely in the frequency function, which is exactly defined on the graph Γ.

Theorem 2 describes the conditions for the vector of the right-hand sides of system (1.7) to be orthogonal to the space of the pairs of functions specified on the graph. These conditions lead to equations describing the change in the topological invariants of the field v along the axis of the vortex, that is, to Eqs (2.1) which relate the pair of functions on the graph Γ. Note that the graph Γ is an example of a well-known object in topology, that is, a so-called Reef graph of Morse's function (see [13], for example). Actually, a Reef graph is a set of level lines of a function, all the critical points of which are non-degenerate. In our case, this function is the stream function of the two-dimensional Euler flow v.

3. PROPERTIES OF THE VORTEX FILAMENT EQUATIONS

Analogy with the Prandtl equations. Self-induced pressure and an analogue of turbulent viscosity. Equations (2.1) are similar to the Prandtl equations in boundary-layer theory: if we put $D = 1$ and assume that $\langle \pi_s \rangle$ is a known function of the single variable s, the first two equations in (2.1) are identical to the Prandtl equations. However, the term $\langle \pi_s \rangle$ cannot, generally speaking, be assumed to be a specified function. It is associated with the Bernoulli function B of the transverse circulations in a vortex. This brings Eqs (2.1) closer to the so-called Prandtl equations with a self-induced pressure (see [14], for example). The same also applies to the function D^2, which appears on the right-hand side of system of equations (2.1), which describe the effect of viscosity on a vortex filament. These are analogous functions in the equations describing turbulent flows (the coefficients of turbulent viscosity, see [15], for example). However, note that, unlike the above-mentioned equations, equalities (2.1) (and, in particular, the expressions for the coefficient D^2 and the "self-induced pressure" $\langle \pi_s \rangle$) are obtained not from physical considerations but as a mathematical consequence of the existence of a solution of the Navier–Stokes equations which describes a vortex filament: the connection between these terms and the function B is described by the Euler equations (1.5).

Kirchhoff's conditions, which are imposed on the function a (together with the null conditions at the vertices of degree 1), enable one to eliminate this function from Eqs (2.1) in exactly the same way as the corresponding function is eliminated from the Prandtl's equations. Actually, it is possible to obtain a formula which expresses a in terms of w on an arbitrary edge of the graph Γ:

$$a(I,s) = -\int_{I_0}^{I} \frac{\partial w}{\partial s}(I', s)dI' - \sum_j \int_{m_j} \frac{\partial w}{\partial s}(I, s)dI$$

where I_0 is the value of the parameter I at the initial point of the edge and summation is carried out over all edges m_j of the graph, which are counteraccessible from the given edge (that is, accessible on moving from the edge in the direction in which the parameter I decreases). On substituting this expression into the first and third equalities of (2.1), we obtain a system of two equations which do not contain the function a.

Additional "boundary" conditions for the equations of a vortex filament. Since Eqs (2.1) in one of the variables are specified on a graph (that is, in a system of segments), it follows that they should be supplemented with conditions at the vertices (at the ends of the segments). Some of these conditions consist of the requirement that the functions B and w should be continuous on the graph Γ. However, generally speaking, this is insufficient and, since second derivatives with respect to the variable I occur in the equations, it is necessary to add the conditions of α-smoothness [16], which are specified at the vertices of the graph and connect the derivatives of an unknown function along the edges which converge at a given vertex.

It is found that, in the case of the equations of a vortex filament, these conditions (and the equations themselves) follow from the existence of an asymptotic. In fact, it has been proved in [10] that, if the functions υ, wC^2 are smooth, then the solutions of the equations of a vortex filament at each vertex of degree 3 satisfy Kirchhoff's conditions

$$\left(D^2\frac{\partial B}{\partial I}\right)_1+\left(D^2\frac{\partial B}{\partial I}\right)_2=\left(D^2\frac{\partial B}{\partial I}\right)_3\quad \left(D^2\frac{\partial w}{\partial I}\right)_1+\left(D^2\frac{\partial w}{\partial I}\right)_2=\left(D^2\frac{\partial w}{\partial I}\right)_3$$

where the subscripts 1 and 2 denote the limits of the corresponding functions at the given vertex with respect to the edges converging at this vertex and the subscript 3 denotes the limit with respect to the emerging edge (the graph is orientated in the direction in which the parameter I increases).

In Eqs (2.1) there are free parameters associated, first, with the freedom in determining the parametrization of the graph Γ and, second, with the freedom in choosing the boundary conditions for w and B at vertices of degree 1.

We will now clearly describe these parameters. Specification of the parametrization of graph Γ is equivalent to the specification of the values of the parameter I at the ends of its edges. Here, at all vertices of degree 1, this value must be equal to zero and at all vertices of degree 3, which correspond to three edges terminating at a given vertex, they must satisfy Kirchhoff's relation $I_3 = I_1 + I_2$. Hence, $2m$ free parameters I_1^j, I_2^j ($j = 1, \ldots, m$) arise in the specification of the parametrization, where m is the number of vertices of degree 3 of the graph Γ (or the number of saddle points of the solenoidal field υ). Moreover, for the graph equations (2.1), the boundary conditions corresponding to the values of w^j, B^j at its vertices of degree 1 can be specified in an arbitrary way (see [16]), that is, at "centre" type singular points of the field υ. A set of $2M$ parameters therefore arises, where M is the total number of singular points of the field υ. It turns out that it is possible to obtain relations connecting these parameters. They follow from the same condition for the equations of the first approximation (1.7) to be solvable. In fact, the following assertion holds.

Assertion 2. Suppose Eqs (1.7) admit of a C^2-smooth solution υ^1, w_1. Then, the relations

$$H(r_j)=0,\quad \operatorname{rot}_z F(r_j)-G(r_j)\operatorname{rot}_z v(r_j)=0 \tag{3.1}$$

are satisfied at each singular point $r_j (j = 1, \ldots, M)$ of the vector field υ.

Proof. Consider the first equality of (1.7) at a singular point r_j. Since $\upsilon(r_j) = 0$, the first term on the left-hand side of this equality vanishes. Next, the function w is constant in the trajectories of the vector field υ and hence, the singular points of υ are the critical points of w. Consequently, the second term on the left-hand side of (1.7) also vanishes, which proves the first equality of (3.1). In order to prove the second equality of (3.1), we apply the operation rot_z to the second (vector) equation of (1.7) and substitute the singular point r_j. into the resulting equality. On taking into account the fact that the function $\operatorname{rot}_z\upsilon$ is constant on the trajectories of υ, after some elementary calculations, we obtain the second equality of (3.1).

Remark. The second equality in (3.1) is a consequence of the existence of a δ-function in the cokernel of the linearized Euler operator. Actually, any vector field of the form

$$\xi = \operatorname{sgrad} \delta(z - r_j)$$

satisfies system (1.8) (which is adjoint to the linearized Euler system), where we mean by a skew-gradient, sgrad, of a δ-function a linear functional in the space of two-dimensional Schwartz vector fields which acts according to the rule

$$\operatorname{sgrad} \delta(z - r_j)[\eta(z)] = -\operatorname{rot} \eta(r_j)$$

Similarly, the first equality in (3.1) is a consequence of the existence of δ-functions in the cokernel of the linearized operator (1.6)

We will now write conditions (3.1) in terms of the functions B, w. The values of these functions at the singular points r_j of the vector field υ are denoted by $B_j(s)$, $w_j(s)$ (note that these values are the critical ones of the above-mentioned functions).

Assertion 3. Equalities (3.1) are equivalent to the relations

$$\frac{\partial B_j(s)}{\partial s} = \nu_0 \Delta . w(r_j)$$

$$q_j(s)\frac{\partial w_j(s)}{\partial s} - w_j(s)\frac{\partial q_j(s)}{\partial s} = -\nu_0 \Delta(\mathrm{rot}_z v)(r_j) \tag{3.2}$$

The value, at a point r_j of the vorticity function of the field v: $q_j(s)$, $\mathrm{rot}_z v(r_j)$ is denoted by $q_j(s)$ (we recall that this function is constant in the trajectories of the field v and, in particular, r_j are its critical points).

Remark. Relations (3.2) are $2M$ relations in the $2M$ free parameters mentioned above in the solutions of the vortex filament equations (2.1).

Proof. Consider the first equality in (3.1). Substituting $H(z, s)$ from formula (1.7) and taking account of the fact that

$$v(r_j) = \nabla_z w(r_j) = 0$$

we obtain

$$w\frac{\partial w}{\partial s}(r_j) - \frac{1}{2}\frac{\partial}{\partial s}W^2 + \frac{\partial \pi_0}{\partial s}(r_j) - \nu_0 \Delta_z w(r_j) = 0 \tag{3.3}$$

Next, it follows from the Euler equations that

$$\frac{\partial \pi_0}{\partial s}(r_j) = \frac{\partial B_0}{\partial s}(r_j)$$

Finally, since r_j is a critical point of the function w, we have the relation

$$2w(r_j)\frac{\partial w}{\partial s}(r_j) = \frac{\partial w_j^2}{\partial s}$$

Substituting the resulting equalities into (3.3), we obtain the first equality in (3.2). We now consider the second relation in (3.1). Substituting F and G from formulae (1.7) into this relation, taking account of the fact that r_j is a critical point of w and using standard formulae of vector analysis, we obtain

$$w_j\frac{\partial q_j}{\partial s} + q_j \,\mathrm{tr}\, A - q_j\frac{\partial}{\partial s}(w(r_j) - W) - \nu_0 \Delta\, \mathrm{rot}_z v\,(r_j) = 0$$

Taking into account the fact that $\mathrm{tr}A = -\partial W/\partial s$, we obtain the second equality in (3.3).

Integral identities and conservation laws. Equations (2.1) possess a number of interesting properties, namely, the solutions of these equations satisfy an infinite number of integral identities (of the type of energy balance equations) which, when $\nu_0 = 0$ (that is, in an ideal fluid) reduce to conservation laws which hold for any value of ν_0.

Theorem 4. Suppose the functions v, π_0, w satisfy the Euler equations (1.5) and (1.6) and the functions w and B, constructed using them, and a certain function a which is smooth in the edges of graph Γ, satisfy Eqs (2.1). Suppose a, B, w satisfy Kirchhoff's conditions and, moreover, the functions w, v, π_0, a_0 decrease when $l \to \infty$ as $O(l^{-1-\delta})$, $\delta > 0$. The identities

$$\frac{\partial}{\partial s}\int_\Gamma (w - W)dl = 0, \quad \frac{\partial}{\partial s}\int_\Gamma \left(w^2 - W^2 - \left\langle\frac{v^2}{2}\right\rangle\right)dl = 0 \tag{3.4}$$

$$\frac{\partial}{\partial s}\int_\Gamma wQ(B)dl = -\nu_0 \int_\Gamma [Q''(B)D^2\left(\frac{\partial B}{\partial l}\right)^2 + \Lambda^2 Q'(B)]dl \tag{3.5}$$

then hold. Here $Q(t)$ is an arbitrary smooth function which vanishes when $t = 0$ and the angular brackets denote averaging along the trajectories of the field v.

The proof was previously obtained in [10].

Remark. Equalities (3.4) are conservation laws for the vortex filament equations. These are laws of conservation of the longitudinal momentum of the vortex (of the integral of the longitudinal component of the velocity over the transverse cross-section) and the conservation of the difference between the energies of the longitudinal flow and the transverse circulations in a vortex. Equalities (3.5) are similar to energy balance equations and the right-hand side describes the viscous dissipation. When $\nu_0 = 0$ (that is, if the fluid is ideal or the viscosity is much smaller than ε^2), Eqs (3.5) also reduce to the conservation laws

$$\frac{\partial}{\partial s}\int_\Gamma wQ(B)dI = 0$$

These qualities recall the well-known series of conservation laws for the two-dimensional, unsteady Euler equations. We recall that, by virtue of the above mentioned equations, the integral of any function (in particular, of any power) of the curl of the velocity field is conserved.

4. A RADIALLY-SYMMETRIC VORTEX FILAMENT

Consider vector fields υ with the simplest phase portrait, that is, assume that there is only a single singular point in the z plane in the case of this field. This point is then necessarily of the "centre" type. In this case, the corresponding graph Γ is the half line $I = (0, \infty)$. An extensive class of such solutions of the Euler equations (1.5) is known. These are radially-symmetric fields of the form

$$\upsilon = (-\partial\psi(r)/\partial z_2, \partial\psi(r)/\partial z_1), \quad r = \sqrt{z_1^2 + z_2^2}$$

where $\Psi(r)$ is a smooth function. Any such vector field satisfies Eqs (1.5) and any function $w(r)$ satisfies Eqs (1.6). No other solutions of the Euler equations with the indicated topology of the phase curves are known and a hypothesis has been proposed in [17] according to which there are no other solutions. In every case, we shall subsequently confine ourselves to considering radially-symmetric solutions. In the radially-symmetric case, the action—angle variables are defined in the whole of the z plane with a single singularity at zero. They have the form $I = r^2/2$, $\varphi = \phi$ (the polar angle). In this situation, it is simpler in the calculations to use the frequency $\omega(I)$ in a trajectory instead of the Bernoulli function B.

We will now write out Eqs (2.1), which describe a vortex filament, for this case. We subtract the first equation in (2.1), multiplied by w, from the last equation in (2.1) and obtain

$$\left(w\frac{\partial}{\partial s} + a\frac{\partial}{\partial I}\right)(I^2\omega^2) = 4\nu_0 I^2\omega\frac{\partial^2}{\partial I^2}(I\omega) \tag{4.1}$$

From the second equation in (2.1), we have $w = \partial\psi/\partial I$, $a = -\partial\psi/\partial s$, where $\psi(I, s)$ is a smooth function (a stream function). On differentiating the first equation in (2.1) with respect to I and writing (4.1) in terms of ψ, we finally obtain the following system of equations, which is equivalent, in the case under consideration, to the equations of the vortex filament

$$\begin{gathered}\{\psi, I^2\omega^2\} = 4\nu_0 I^2\omega\frac{\partial^2}{\partial I^2}(I\omega)\\ \frac{\partial\omega^2}{\partial s} + \left\{\psi, \frac{\partial^2\psi}{\partial I^2}\right\} = 2\nu_0\frac{\partial^2}{\partial I^2}\left(I\frac{\partial^2\psi}{\partial I^2}\right)\end{gathered} \tag{4.2}$$

where $\{,\}$ is a Poisson bracket. Conditions (3.2) lead, after some simple calculations, to the two boundary conditions when $I = 0$:

$$\frac{\partial}{\partial s}\left(\left(\frac{\partial\psi}{\partial I}\right)^2 - W^2 - \int_0^\infty\omega^2 dI\right) = 2\nu_0\frac{\partial^2\psi}{\partial I^2}, \quad \frac{\partial}{\partial s}\left(\frac{1}{\omega}\frac{\partial\psi}{\partial I}\right) = 4\nu_0\frac{\partial}{\partial I}\frac{1}{\omega} \tag{4.3}$$

Two special cases of Eqs (4.2) are considered next.

A radially-symmetric vortex in an ideal fluid. Suppose that $\nu_0 = 0$. Then, the equations of system (4.2) have zero right-hand sides. In this case, the first equality is satisfied, if $\omega^2 = f_0(\Psi)/I^2$ for a certain smooth function f_0. After substituting this expression into the second equality, it is transformed to the form

$$\{\psi, \partial^2\psi/\partial I^2 + f_0'(\psi)/I\} = 0$$

The last equality will obviously be satisfied if

$$\partial^2\psi/\partial I^2 + f_0'(\psi)/I = -g(\psi)$$

for a certain smooth function g. Hence, in the radially-symmetric case, the equations of a vortex filament reduce to the "ordinary differential equation" (compare with the Grade-Shafranov equation [18])

$$I d^2\psi/dI^2 + f(\psi) + Ig(\psi) = 0$$

where f and g are arbitrary smooth functions ($f = f_0'$). This equation must be supplemented with the condition for the behaviour of ψ at infinity

$$\psi = W(s)I + \xi(s) + O(I^{-1})$$

and, also, by conditions (4.3) which take the form

$$\psi = \lambda I + O(I^2) \text{ when } I \to 0, \quad \frac{\partial}{\partial s}\left(\int_0^\infty \frac{f_0(\psi)}{I^2} dI - W^2\right) = 0$$

Here λ is a constant (which is independent of both I and s).

A Moffatt–Kida–Ohkitani vortex. We will now consider a radially-symmetric, elongated vortex in the special case of a linear external flow $V = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$. We select the straight line $x_1 = x_2 = 0$ as the trajectory γ. System (4.2) then admits of a simple solution for which

$$\psi = I\lambda_3 s, \quad \partial\omega/\partial s = 0, \quad -I\lambda_3\partial(I^2\omega^2)/\partial I = 4\nu_0 I^2\omega\partial^2(I\omega)/\partial I^2$$

From the last equality, we find

$$\omega = C(1 - \exp(-\beta I))/I = 2C(1 - \exp(-\beta r^2/2))/r^2$$
$$C = \text{const}, \quad \beta = \lambda_3(2\nu_0)^{-1}$$

This solution (together with a correction to it) was previously obtained and investigated in detail in [17].

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